

Relativized Propositional Calculus

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Motivation:

Complexity lower bounds and independence results are easier in a relativized setting. It seems reasonable to define a relativized setting for the propositional calculus in order to prove lower bounds.

Syntax:

The language of **PC(R)** (propositional calculus relativized to R) consists of formulas built from atoms p, q, r, \dots using the usual connectives $0, 1, \wedge, \vee, \neg$, together with the relation symbol R . The usual formation rules for formulas apply, but in addition we agree that

if A_1, \dots, A_n are formulas, $n \geq 0$, then $R(A_1, \dots, A_n)$ is a formula

Semantics:

A structure τ consists of an assignment of a truth value p^τ in $\{0, 1\}$ (where $1 = \text{TRUE}$ and $0 = \text{FALSE}$) to each atom p , together with a set $R^\tau \subseteq \{0, 1\}^*$ of binary strings. Then each formula A of **PC(R)** gets a truth value $A^\tau \in \{0, 1\}$ in the obvious way. In particular,

$$R(A_1, \dots, A_n)^\tau = 1 \iff A_1^\tau \dots A_n^\tau \in R^\tau$$

This syntax and semantics is essentially the same as that defined by Ben-David and Gringauze [3]. (See also [1].)

We say that A is *valid* iff $A^\tau = 1$ for all structures τ , and A is *satisfiable* iff $A^\tau = 1$ for some structure τ .

For example,

$$(R(p) \wedge R(\neg p)) \supset (R(q) \vee R(\neg q))$$

is valid. In general, A is valid iff $\neg A$ is unsatisfiable.

Theorem 1: The satisfiability problem for **PC(R)** formulas is in **NP** (and hence **NP**-complete).

Proof: A certificate for satisfiability need only specify τ for each atom in A , and for each occurrence of the form $R(B_1, \dots, B_n)$ in A , some string $v_1 \dots v_n \in \{0, 1\}^n$ is specified to either be in R^τ or not in R^τ . \square

System PK(R):

This is Gentzen's sequent system **PK** for the propositional calculus (see for example [4] or [5]), except formulas are allowed to be **PC(R)** formulas, and in addition to the axiom scheme $A \rightarrow A$, and the axioms $\rightarrow 1$ and $0 \rightarrow$, we add the axiom scheme

¹This is a slight revision of a working paper from June 4, 2003. Much of this material was presented at the complexity theory workshop at Overwolfach, 30 April, 2003.

AX : $\neg A \vee B, A \vee \neg B, R(\vec{C}, A, \vec{D}) \rightarrow R(\vec{C}, B, \vec{D})$

which asserts that if A and B are equivalent, then one can be substituted for the other as an argument of R .

Using **AX**, each of the following four schemes E1,E2,E3,E4 has a **PK(R)** proof with a constant number of sequents:

E1) $A, R(\vec{C}, A, \vec{D}) \rightarrow R(\vec{C}, 1, \vec{D})$

E2) $A, R(\vec{C}, 1, \vec{D}) \rightarrow R(\vec{C}, A, \vec{D})$

E3) $R(\vec{C}, A, \vec{D}) \rightarrow A, R(\vec{C}, 0, \vec{D})$

E4) $R(\vec{C}, 0, \vec{D}) \rightarrow A, R(\vec{C}, A, \vec{D})$

Theorem 2: **PK(R)** is sound and complete. Further every valid sequent S has a **PK(R)** proof π with $O(2^{|S|})$ sequents, where each sequent in π has length $O(|S|)$, where $|S|$ is the total number of symbols in S .

Remark: In counting the number of sequents in a proof, we do not count weakenings and exchanges.

Proof: Soundness asserts that every sequent derivable in **PK(R)** is valid. This is true because the axioms are valid and the rules preserve validity.

Completeness asserts that every valid sequent has a **PK(R)** proof. To get an upper bound on the number of lines in the proof, we make the following definition:

Definition 1: The cost $c(A)$ of a formula A is the number of occurrences of \wedge, \vee, \neg in A plus, for each subformula $R(B_1, \dots, B_n)$ in A , the number of formulas in the sequence B_1, \dots, B_n other than 0 or 1. The cost $c(S)$ of a sequent S is the sum of the costs of the formulas in the sequent.

For example, the cost of $R(p \wedge q, p \wedge q, p, 0, 1, 1)$ is 5: 2 for the two occurrences of \wedge , and 3 for the three nontrivial arguments of R .

Note that $c(A) \leq |A|$, where $|A|$ is the number of symbols in A , counting commas.

Lemma 1: For some constant d , each valid sequent S has a **PK(R)** proof with at most $d2^{c(S)}$ lines, where each line has length $O(|S|)$.

Proof: Induction on $c(S)$.

Basis: Suppose that $c(\Gamma \rightarrow \Delta) = 0$ and $\Gamma \rightarrow \Delta$ is valid. Then any occurrence of R must be as a formula of the form $R(v_1, \dots, v_n)$ in one of the sequences Γ or Δ , where each v_i is either 0 or 1. It is easy to check that either 1 is a formula in Δ , or 0 is a formula in Γ , or some formula occurs in both Γ and Δ . In each case, $\Gamma \rightarrow \Delta$ can be derived from an axiom (other than **AX**) by weakenings and exchanges.

Induction Step: $c(\Gamma \rightarrow \Delta) > 0$. Then some formula in either Γ or Δ must have a principal connective that is either \wedge, \vee, \neg or R . For the cases \wedge, \vee, \neg we derive $\Gamma \rightarrow \Delta$ by

the appropriate **PK** introduction rule (left or right), thus reducing the problem to deriving one or two valid sequents, each of reduced cost, so the Induction Hypothesis applies.

Now suppose that $\Gamma \rightarrow \Delta$ has the form

$$\Gamma' \rightarrow \Delta', R(\vec{C}, A, \vec{D}) \quad (1)$$

where A is not 0 or 1. Then we use the derivation below, based on E2 and E4 above, where we have omitted weakenings and exchanges. All indicated inferences use the **cut** rule.

$$\frac{\frac{A, \Gamma' \rightarrow \Delta', R(\vec{C}, 1, \vec{D}) \quad E2}{A, \Gamma' \rightarrow \Delta', R(\vec{C}, A, \vec{D})} \quad \frac{\Gamma' \rightarrow \Delta', A, R(\vec{C}, 0, \vec{D}) \quad E4}{\Gamma' \rightarrow \Delta', A, R(\vec{C}, A, \vec{D})}}{\Gamma' \rightarrow \Delta', R(\vec{C}, A, \vec{D})}$$

This reduces the proof of (1) to the proof of two valid sequents, each of which has cost one less than the cost of (1). The induction hypothesis applied to these two sequents gives us the desired result.

The remaining case to consider is that $\Gamma \rightarrow \Delta$ has the form

$$\Gamma', R(\vec{C}, A, \vec{D}) \rightarrow \Delta'$$

where again A is not 0 or 1. This time we use the derivation below, using E1 and E3:

$$\frac{\frac{A, R(\vec{C}, 1, \vec{D}), \Gamma' \rightarrow \Delta' \quad E1}{A, R(\vec{C}, A, \vec{D}), \Gamma' \rightarrow \Delta'} \quad \frac{R(\vec{C}, 0, \vec{D}), \Gamma' \rightarrow \Delta', A \quad E3}{R(\vec{C}, A, \vec{D}), \Gamma' \rightarrow \Delta', A}}{\Gamma', R(\vec{C}, A, \vec{D}) \rightarrow \Delta'}$$

Now we apply the induction hypothesis, as in the previous case.

Quantified Relativized Propositional Calculus

Formulas in **QPC(R)** are like those in **PC(R)**, except we now allow quantifiers $\forall x$ and $\exists x$, for an atom x . The semantics are obtained in the obvious way by letting x range over $\{0, 1\}$.

Notation: $\Pi_1^q(R)$ is the class of formulas of **QPC(R)** of the form

$$\forall \vec{x} A(\vec{x}, \vec{p}, R)$$

where A is quantifier-free.

Theorem 3: The satisfiability problem for **QPC(R)** is complete for **NEXP**. The same is true for the satisfiability problem restricted to $\Pi_1^q(R)$ formulas.

Proof: It is easy to see that the satisfiability problem is in **NEXP**: Given a formula A of **QPC(R)**, let n be the largest number of arguments of any occurrence of R in A . Guess at

a structure τ for A by writing down truth values to the free variables of A , and specifying R^τ for R up to strings of length n by writing down a subset of $\{0, 1\}^{\leq n}$. Now verify that τ satisfies A .

Hardness can be established either by a direct reduction of Turing machine computations to **QPC(R)** satisfiability (proof due to Charles Rackoff), or by using the proof that succinct circuit satisfiability is NEXP complete (see page 494 of Christos Papadimitriou's textbook on Computational Complexity) (proof due to Tsuyoshi Morioka).

Notation: $|A|$ denotes the length of a formula A ; that is, the total number of occurrences of symbols in A .

Note that if there are many different variables occurring in A then the binary length of A could be as more like $|A| \log |A|$.

Lemma 1A: (with Rackoff) For every nondeterministic TM M there is a polytime transformation F_M such that for all $x \in \{0, 1\}^*$, $F_M(x)$ is a $\Pi_1^q(R)$ formula, and $|F_M(x)| = O(|x|)$, and

$$A = F_M(x) \text{ is satisfiable} \iff M \text{ accepts } x \text{ in at most } 2^{|x|} \text{ steps}$$

Proof Outline: The proof is like that of the Cook-Levin Theorem. Let C_0, C_2, \dots, C_T be a computation of $T = 2^n$ steps of M on input x , where $n = |x|$. Here C_i is a bit string of length $O(2^n)$ coding the configuration of M at step i . Thus the computation can be represented by a relation $R \subseteq \{0, 1\}^{O(n)}$, where $R(\vec{p}, \vec{q})$ represents bit \vec{p} of $C_{\vec{q}}$.

Then $F_M(x)$ is the prenex form of $S \wedge I \wedge E$ where

S asserts that the computation starts right

I asserts that the computation increments right

E asserts that the computation ends right

The formula E is easy, since it merely asserts that the configuration C_T is in an accepting state.

The formula S asserts that the initial configuration, coded by $R(\vec{p}, \vec{0})$ (as \vec{p} ranges over all possible values), represents a tape configuration consisting of x followed by blanks, and the initial state.

To see how to express this with a formula of length $O(n)$ we assume for simplicity that $x = x_1 \dots x_n$ is a bit string over $\{0, 1\}$. We show how to construct a formula $S_1(\vec{p})$ of length $O(n)$ which asserts that for $i = 1, \dots, n$ if \vec{p} represents i in binary then $(R(\vec{p}, \vec{0}) \leftrightarrow x_i)$. This explains the interesting part of the construction of S .

To see how to construct S_1 , let $k = \lceil \log_2(n + 1) \rceil$ and suppose p_1, \dots, p_k represent the k low-order bits when \vec{p} represents a binary number i . (When $i \leq n$, then the remaining bits p_{k+1}, \dots, p_n are 0.) Consider a Boolean circuit α with inputs p_1, \dots, p_k and outputs r_1, \dots, r_n such that $r_i = 1$ iff $p_1 \dots p_k$ represents i in binary. Note that α can be constructed with $O(n)$ gates by a simple recursion on k .

Let $\beta(p_1, \dots, p_k, \vec{g}, \vec{r})$ be a (quantifier-free) propositional formula of length $O(n)$ which holds iff the circuit α with input values p_1, \dots, p_k takes on values \vec{g} for its internal gates and values \vec{r} for its output gates. Then $S_1(\vec{p})$ is the formula

$$\forall \vec{g} \forall \vec{r} [(\beta(p_1, \dots, p_k, \vec{g}, \vec{r}) \wedge Z(\vec{p})) \rightarrow [R(\vec{p}, \vec{0}) \leftrightarrow ((r_1 \wedge x_1) \vee \dots \vee (r_n \wedge x_n))]]$$

where

$$Z(\vec{p}) \equiv \neg p_{k+1} \wedge \dots \wedge \neg p_{cn}$$

It remains to discuss the formula I . This asserts that for all $t < 2^n$, C_{t+1} is the successor configuration to C_t (when C_t and C_{t+1} are represented by R .) Given a reasonable representation of the Turing machine configurations, it is straightforward to construct such a $\Pi_1^q(R)$ -formula I of length $O(n)$. \square

Corollary: There is no proof system for the valid formulas of **QPC(R)** (or for the valid $\Sigma_1^q(R)$ formulas) with the property that every valid formula A has a proof P such that

$$|P| = 2^{o(|A|)} \quad (2)$$

where $|P|$ is the bit length of P .

Proof of the Corollary: We use the following

Fact: There is a universal nondeterministic TM M_0 such that for every nondeterministic TM M and all sufficiently large strings x which code M ,

$$M_0 \text{ accepts } x \text{ within } 2^{|x|} \text{ steps} \iff M \text{ accepts } x \text{ within } 2^{0.4|x|} \text{ steps}$$

Let c_0 be a constant such that, referring to Lemma 1A,

$$|F_{M_0}(x)| \leq c_0|x|, \text{ for all sufficiently long } x$$

Now suppose Π is a proof system for unsatisfiability which violates the Corollary, so every unsatisfiable $\Sigma_1^q(R)$ formula A has a proof P satisfying (2). Let M_1 be a nondeterministic TM which on input x computes $A = F_{M_0}(x)$, guesses a proof P , and accepts iff P is a Π proof of A (showing that A is unsatisfiable). Let $\delta = 0.2/c_0$ and let x_1 be a sufficiently long string coding M_1 . Then

$$\begin{aligned} & M_1 \text{ accepts } x_1 \text{ within } 2^{0.4|x_1|} \text{ steps} \\ \iff & \text{there is a } \Pi \text{ proof } P \text{ of } A = F_{M_0}(x_1) \text{ where } |A| \leq c_0|x_1| \text{ and } |P| \leq 2^{\delta|A|} \leq 2^{0.2|x_1|} \\ \iff & A \text{ is unsatisfiable} \\ \iff & M_0 \text{ does not accept } x_1 \text{ within } 2^{|x_1|} \text{ steps} \\ \iff & M_1 \text{ does not accept } x_1 \text{ within } 2^{0.4|x_1|} \text{ steps.} \end{aligned}$$

This is a contradiction. \square

System G(R):

This is the system G of quantified propositional calculus described in section 4.6 of Krajicek's

book [5], extended so that formulas are allowed to be **QPC(R)** formulas, and we allow the axiom scheme **AX** above. In other words, **G(R)** is obtained from **PK(R)** by extending the definition of formula, and allowing the four quantifier rules of **LK** (Krajicek, page 58).

Theorem 4: **G(R)** is sound and complete.

Proof: Soundness is easy, since as before the axioms are valid and the rules preserve validity.

We prove that every valid sequent has a **G(R)** proof by double induction, first on the maximum quantifier depth of formulas in the sequent, and second on the cost $c(S)$ of the sequent, as defined in Definition 1 above.

To see how to reduce the quantifier depth, consider the case

$$\Gamma' \rightarrow \Delta', \exists x A(x)$$

This can be derived by two applications of \exists -**right** and one of contraction from

$$\Gamma' \rightarrow \Delta', A(0), A(1)$$

and this sequent is valid if the previous one is valid. \square

Remark: It seems that the obvious upper bound for the above proof length is doubly exponential, even in the case of nonrelativized G , and even for the case nonrelativized G_1 .

Consider the example

$$\rightarrow \exists x_1 \dots \exists x_n (A_1 \wedge \dots \wedge A_m)$$

If we apply the above method to get rid of the existential quantifiers, we obtain a sequent with 2^n formulas, each of which is a conjunction of m formulas. Now to unwind all of these conjunctions in the usual way seems to generate 2^{m2^n} sequents.

Rackoff points out that this large upper bound is not surprising for the relativized case. In fact, if a simply exponential upper bound could be found, it would follow from Theorem 3 that **NEXP** = **coNEXP**.

However there is a simply exponential upper bound for the nonrelativized case.

Theorem 5: (See Theorem VII.3.9 in [4].) Every valid sequent S of **QPC** (with no R) has a tree-like **G** proof with $O(2^{|S|})$ sequents (not counting weakenings and exchanges), where each sequent has length $O(|S|)$ and all cut formulas are atomic.

Work to be done:

- Carry out the translations of the relativized theories $S_2^i(R)$ and $T_2^i(R)$ into **QPC(R)**. It may be easier to translate the two-sorted versions $V^i(R)$ and $TV^i(R)$. (The theories V^i and TV^i are presented in [4], where propositional translations are given.)
- Once the translations have been written down, it should be possible to describe families of valid **QPC(R)** formulas corresponding to various search problems, and prove lower bounds

on their $\mathbf{G}(\mathbf{R})$ proof lengths by the same search problem separations used to separate various relativized theories of bounded arithmetic.

Example: Let $\mathbf{WPHP}(R, n)$ be a relativized propositional formula (in fact a $\Sigma_2^q(R)$ formula) representing the weak pigeonhole principle $\mathbf{PHP}_a^{a^2}$ as follows. (Here we assume that \vec{p} and \vec{q} are vectors of $2n$ variables, while \vec{r} and \vec{s} are vectors of just n variables.)

$$\mathbf{WPHP}(R, n) \equiv \exists \vec{p} \exists \vec{q} \exists \vec{r} [(\vec{p} \neq \vec{q} \wedge R(\vec{p}, \vec{r}) \wedge R(\vec{q}, \vec{r})) \vee \forall \vec{s} \neg R(\vec{p}, \vec{s})]$$

Conjecture 1: $\langle \mathbf{WPHP}(R, n) \rangle$ does not have polysize $\mathbf{G}_2^*(\mathbf{R})$ proofs.

Proposed Proof Outline:

(i) Theorem 11.3.1, page 220 of Krajicek's book shows that the witnessing problem for $\mathbf{WPHP}(R, n)$ is not in $FP^{NP(R)}$.

(ii) The witnessing problem for $\mathbf{G}_2^*(\mathbf{R})$ proofs of $\Sigma_2^q(R)$ formulas is in $FP^{NP(R)}$. This is by analogy with the fact that the witnessing problem for $\mathbf{G}_1^*(\mathbf{R})$ proofs of $\Sigma_1^q(R)$ formulas is in $FP(R)$.

(iii) If $\langle \mathbf{WPHP}(R, n) \rangle$ has polysize $\mathbf{G}_2^*(\mathbf{R})$ proofs, then given n we could use an NP oracle to find a proof of $\mathbf{WPHP}(R, n)$, and then use (ii) to solve the witnessing problem with an $NP(R)$ oracle. This contradicts (i). \square

In the same vein, we know (by translations into bounded depth Frege systems) that

$$S_2(R) \not\vdash PHP(R)$$

(see Pitassi's thesis). This suggests

Conjecture 2: $\langle \mathbf{PHP}(R, n) \rangle$ does not have polysize $\mathbf{G}_1(\mathbf{R})$ -proofs, for any i .

Apparently we can translate theorems of $S_2(R)$ both into quasipolysize families of bounded depth Frege proofs, and into polysize families of $\mathbf{G}(\mathbf{R})$ proofs. This leads to

Conjecture 3 (Pudlak): Find an RSUV style isomorphism between AC^0 -Frege and $\mathbf{G}(\mathbf{R})$.

In a slightly different vein, we have

Conjecture 4: (Morioka:) The $\mathbf{ITER}(\mathbf{R})$ Tautologies do not have polysize $\mathbf{G}_1^*(\mathbf{R})$ proofs.

Proposed Proof (Morioka): Prove a superpolynomial lower bound for the circuit size for solving $\mathbf{ITER}(\mathbf{R})$.

- Think about using the oracle separations of \mathbf{NC} and \mathbf{P} in [2] to separate relativized $\mathbf{G}_1^*(\mathbf{R})$ and $\mathbf{G}_1(\mathbf{R})$.

- (Far out:) Try for lower bounds for unrelativized \mathbf{G} . Of course there's no super proof system for \mathbf{QPC} (including \mathbf{G}) under the assumption $\mathbf{NP} \neq \mathbf{PSPACE}$. Can we get a lower bound for \mathbf{G} proofs under the weaker assumption $\mathbf{P} \neq \mathbf{PSPACE}$?

References

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